Vector Algebra using Orthonormal Base Vectors

Q: *Just why do we express a vector in terms of 3 orthonormal base vectors? Doesn't this just make things even more complicated ??*

A: Actually, it makes things **much** simpler. The **evaluation** of vector operations such as addition, subtraction, multiplication, dot product, and cross product all become straightforward **if** all vectors are expressed using the **same** set of base vectors.

Consider two vectors **A** and **B**, each expressed using the same set of base vectors $\hat{a}_x, \hat{a}_y, \hat{a}_z$:

$$\mathbf{A} = A_x\, \hat{a}_x + A_y\, \hat{a}_y + A_z\, \hat{a}_z$$

$$\mathbf{B} = B_x\, \hat{a}_x + B_y\, \hat{a}_y + B_z\, \hat{a}_z$$

1. *Addition and Subtraction*

If we **add** these two vectors together, we find:

$$\begin{aligned}\mathbf{A}+\mathbf{B} &= \left(A_x\,\hat{a}_x + A_y\,\hat{a}_y + A_z\,\hat{a}_z\right) + \left(B_x\,\hat{a}_x + B_y\,\hat{a}_y + B_z\,\hat{a}_z\right)\\ &= A_x\,\hat{a}_x + B_x\,\hat{a}_x + A_y\,\hat{a}_y + B_y\,\hat{a}_y + A_z\,\hat{a}_z + B_z\,\hat{a}_z\\ &= \left(A_x + B_x\right)\hat{a}_x + \left(A_y + B_y\right)\hat{a}_y + \left(A_z + B_z\right)\hat{a}_z\end{aligned}$$

In other words, each component of the **sum** of two vectors is equal to the sum of each **component**.

Similarly, we find for **subtraction**:

$$\begin{aligned}\mathbf{A}-\mathbf{B} &= \left(A_x\,\hat{a}_x + A_y\,\hat{a}_y + A_z\,\hat{a}_z\right) - \left(B_x\,\hat{a}_x + B_y\,\hat{a}_y + B_z\,\hat{a}_z\right)\\ &= A_x\,\hat{a}_x - B_x\,\hat{a}_x + A_y\,\hat{a}_y - B_y\,\hat{a}_y + A_z\,\hat{a}_z - B_z\,\hat{a}_z\\ &= \left(A_x - B_x\right)\hat{a}_x + \left(A_y - B_y\right)\hat{a}_y + \left(A_z - B_z\right)\hat{a}_z\end{aligned}$$

2. *Vector/Scalar Multiplication*

Say we multiply a scalar a and a vector **B**, i.e., a**B**:

$$
\begin{aligned}
a\mathbf{B} &= a\left(B_x\,\hat{a}_x + B_y\,\hat{a}_y + B_z\,\hat{a}_z\right)\\
&= aB_x\,\hat{a}_x + aB_y\,\hat{a}_y + aB_z\,\hat{a}_z\\
&= \left(aB_x\right)\hat{a}_x + \left(aB_y\right)\hat{a}_y + \left(aB_z\right)\hat{a}_z
\end{aligned}
$$

In other words, each component of the product of a scalar and a vector are equal to the product of the scalar and each component.

3. *Dot Product*

Say we take the **dot product** of **A** and **B**:

$$
\begin{aligned}
\mathbf{A}\cdot\mathbf{B} &= \left(A_x\,\hat{a}_x + A_y\,\hat{a}_y + A_z\,\hat{a}_z\right)\cdot\left(B_x\,\hat{a}_x + B_y\,\hat{a}_y + B_z\,\hat{a}_z\right)\\
&= A_x\,\hat{a}_x\cdot\left(B_x\,\hat{a}_x + B_y\,\hat{a}_y + B_z\,\hat{a}_z\right)\\
&\quad + A_y\,\hat{a}_y\cdot\left(B_x\,\hat{a}_x + B_y\,\hat{a}_y + B_z\,\hat{a}_z\right)\\
&\quad + A_z\,\hat{a}_z\cdot\left(B_x\,\hat{a}_x + B_y\,\hat{a}_y + B_z\,\hat{a}_z\right)\\
&= A_x\,B_x\left(\hat{a}_x\cdot\hat{a}_x\right) + A_x\,B_y\left(\hat{a}_x\cdot\hat{a}_y\right) + A_x\,B_z\left(\hat{a}_x\cdot\hat{a}_z\right)\\
&\quad + A_y\,B_x\left(\hat{a}_y\cdot\hat{a}_x\right) + A_y\,B_y\left(\hat{a}_y\cdot\hat{a}_y\right) + A_y\,B_y\left(\hat{a}_y\cdot\hat{a}_z\right)\\
&\quad + A_z\,B_x\left(\hat{a}_z\cdot\hat{a}_x\right) + A_z\,B_y\left(\hat{a}_z\cdot\hat{a}_y\right) + A_z\,B_z\left(\hat{a}_z\cdot\hat{a}_z\right)
\end{aligned}
$$

A: Be patient! Recall that these are **orthonormal** base vectors, therefore:

$$
\hat{a}_x\cdot\hat{a}_x = \hat{a}_y\cdot\hat{a}_y = \hat{a}_z\cdot\hat{a}_z = 1 \quad \text{and} \quad \hat{a}_x\cdot\hat{a}_y = \hat{a}_y\cdot\hat{a}_z = \hat{a}_z\cdot\hat{a}_x = 0
$$

As a result, our **dot product** expression reduces to this simple expression:

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$$

We can apply this to the expression for determining the **magnitude** of a vector:

$$|\mathbf{A}|^2 = \mathbf{A} \cdot \mathbf{A} = A_x^2 + A_y^2 + A_z^2$$

Therefore:

$$|\mathbf{A}| = \sqrt{\mathbf{A} \cdot \mathbf{A}} = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

For example, consider a previous handout, where we expressed a vector using two different sets of basis vectors:

$$\mathbf{A} = 2.0\hat{a}_x + 1.5\hat{a}_y$$

or,

$$\mathbf{A} = 2.5\hat{b}_y$$

Therefore, the magnitude of **A** is determined to be:

$$|\mathbf{A}| = \sqrt{1.5^2 + 2.0^2} = \sqrt{6.25} = 2.5$$

or,

$$|\mathbf{A}| = \sqrt{2.5^2} = \sqrt{6.25} = 2.5$$

Q: *Hey! We get the **same** answer from both expressions; is this a **coincidence** ?*

A: No! Remember, both expressions represent the **same** vector, only using different sets of base vectors. The magnitude of vector **A** is 2.5, **regardless** of how we choose to express **A**.

4. Cross Product

Now lets take the cross product **A**x**B**:

$$\begin{aligned}
\mathbf{A}\times\mathbf{B} &= \left(A_x\,\hat{a}_x + A_y\,\hat{a}_y + A_z\,\hat{a}_z\right)\times\left(B_x\,\hat{a}_x + B_y\,\hat{a}_y + B_z\,\hat{a}_z\right)\\
&= A_x\,\hat{a}_x \times\left(B_x\,\hat{a}_x + B_y\,\hat{a}_y + B_z\,\hat{a}_z\right)\\
&\quad + A_y\,\hat{a}_y \times\left(B_x\,\hat{a}_x + B_y\,\hat{a}_y + B_z\,\hat{a}_z\right)\\
&\quad + A_z\,\hat{a}_z \times\left(B_x\,\hat{a}_x + B_y\,\hat{a}_y + B_z\,\hat{a}_z\right)\\
&= A_xB_x\left(\hat{a}_x\times\hat{a}_x\right) + A_xB_y\left(\hat{a}_x\times\hat{a}_y\right) + A_xB_z\left(\hat{a}_x\times\hat{a}_z\right)\\
&\quad + A_yB_x\left(\hat{a}_y\times\hat{a}_x\right) + A_yB_y\left(\hat{a}_y\times\hat{a}_y\right) + A_yB_z\left(\hat{a}_y\times\hat{a}_z\right)\\
&\quad + A_zB_x\left(\hat{a}_z\times\hat{a}_x\right) + A_zB_y\left(\hat{a}_z\times\hat{a}_y\right) + A_zB_z\left(\hat{a}_z\times\hat{a}_z\right)
\end{aligned}$$

Remember, we know that:

$$\hat{a}_x \times \hat{a}_x = \hat{a}_y \times \hat{a}_y = \hat{a}_z \times \hat{a}_z = 0$$

also, since base vectors form a **right-handed** system:

$$\hat{a}_x \times \hat{a}_y = \hat{a}_z \qquad \hat{a}_y \times \hat{a}_z = \hat{a}_x \qquad \hat{a}_z \times \hat{a}_x = \hat{a}_y$$

Remember also that **A**x**B** = -(**B**x**A**), therefore:

$$\hat{a}_y \times \hat{a}_x = -\hat{a}_z \qquad \hat{a}_z \times \hat{a}_y = -\hat{a}_x \qquad \hat{a}_x \times \hat{a}_z = -\hat{a}_y$$

Combining all the equations above, we get:

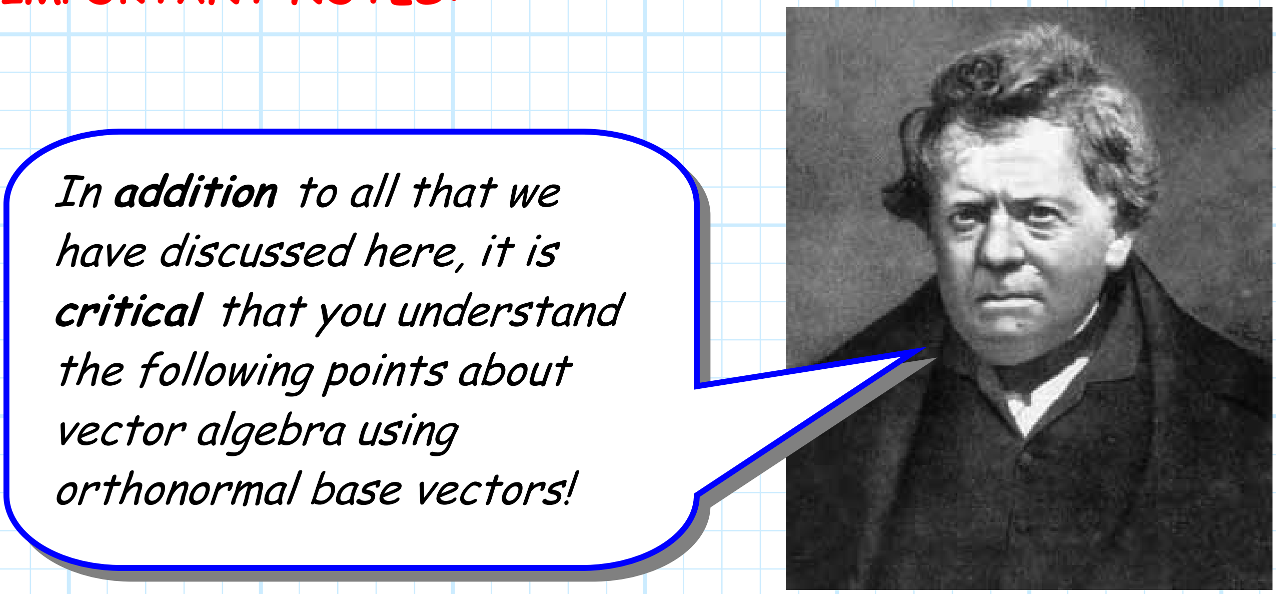
$$\mathbf{A}\times\mathbf{B} = \left(A_y B_Z - A_Z B_y\right)\hat{a}_X + \left(A_Z B_X - A_X B_Z\right)\hat{a}_y + \left(A_X B_y - A_y B_X\right)\hat{a}_Z$$

5. *Triple Product*

Combining the results of the dot product and the cross product, we find that the **triple product** can be expressed as:

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \left(A_x B_y C_z + A_y B_z C_x + A_z B_x C_y\right) - \left(A_x B_z C_y + A_y B_x C_z + A_z B_y C_x\right)$$

IMPORTANT NOTES:



* The results provided in this handout were given for **Cartesian** base vectors ($\hat{a}_x, \hat{a}_y, \hat{a}_z$). However, they are equally valid for **any** right-handed set of base vectors $\hat{a}_1, \hat{a}_2, \hat{a}_3$ (e.g., $\hat{a}_\rho, \hat{a}_\phi, \hat{a}_z$ or $\hat{a}_r, \hat{a}_\theta, \hat{a}_\phi$).

* These results are **algorithms** for evaluating various vector algebraic operations. They are **not** definitions of the operations. The **definitions** of these operations were covered in **Section 2-3**.

* The scalar components A_x, A_y, and A_z represent **either** discrete scalar (e.g., $A_x = 4.2$) **or** scalar field quantities (e.g., $A_\theta = r^2 \sin\theta \cos\phi$.